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STEADY MODE III CRACK PROPAGATION FOLLOWED BY NON-STEADY GROWTH

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Abstract—It is assumed that a mode III crack moves under initially steady-state conditions. At a certain time instant material inhomogenities induce non-uniform crack growth. The stress-intensity factor during a short time interval after the first velocity change is calculated. This is achieved by appropriate superpositioning of known results. A governing equation for the non-steady growth is derived. As an example the clamped infinite strip is analysed. Finally, possible applications of the given results are discussed.

INTRODUCTION

The growing interest in crack arrest problems has stimulated the research of finding analytical and numerical solutions to crack propagation problems. As the arrest process is a case where the crack propagation velocity changes its magnitude more or less rapidly, it is evident that solutions for non-uniform crack motion are needed. Particular classes of such problems have been solved analytically. In these cases it is assumed that the crack moves in an unbounded medium. These solutions can be applied also to finite bodies during the time-interval before interaction with the boundaries or the other tip occurs. This limits the range of validity to a rather short time after the beginning of crack growth. For mode III-problems, the first solution was given by Kostrov[1]. Further extensions have been made by Eshelby[2] among others[3, 4]. The corresponding mode-I problems were solved by Freund[5–8]. A review of methods for solving these problems has recently been published[15].

Obviously the applicability of such solutions to crack arrest problems is limited, since in general a substantial amount of crack growth has taken place before the arrest. An approximation which often is fairly good, is that the crack has propagated steadily before the arrest. Recently, some idealized problems of this kind have been considered[9, 10]. Here it is assumed that the crack initially moves under steady-state conditions and then stops momentaneously. In [9] the short time behaviour of mode-I cases was treated. A particular mode III problem was solved in [10], where also the long-time behaviour was considered. In these papers the conditions for a momentaneous arrest of the assumed kind were discussed.

It is clear that in many cases the assumption of a momentaneous arrest is too crude. We will therefore in the present paper consider an extension of the mode III results to the case where an arbitrary growth after the steady phase is allowed. An analytical procedure, however, does only seem possible for the short-time behaviour.

STATEMENT OF THE PROBLEM

Consider a crack propagating under steady mode III conditions along the x-axis of a fixed (x, y)-coordinate system (Fig. 1a). In this figure, displacements and stresses have for clarity been drawn as for the mode I case. Let w denote the only nonvanishing displacement component. On the outer boundaries some conditions are specified. Cut the body along the x-axis and consider the upper half. Introduce a moving coordinate system attached to the crack tip (η, ξ). Along the crack plane we then have, if the propagation velocity is V.

$$\text{On } y = 0: \quad \tau_{yz} = 0 \qquad x - Vt = \eta < 0 \tag{1}$$

$$\tau_{yz} = \tau^s(\eta) \qquad x - Vt = \eta > 0 \tag{2}$$

$$w = w^s(\eta) \qquad x - Vt = \eta < 0 \tag{3}$$

$$w = 0 \qquad x - Vt = \eta > 0. \tag{4}$$

It is assumed that the solution to this problem (A) is known and in particular the functions $\tau^s(\eta)$ and $w^s(\eta)$.

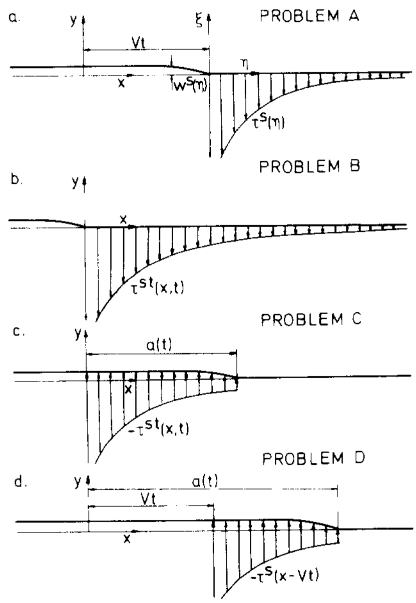

Fig. 1. Illustration of the different boundary value problems.

Suppose now that at $t = 0$, the crack velocity changes in some arbitrary way. The object of the present analysis is to determine the stress-intensity factor $K(t)$ after the first velocity change. The validity of the solution will be limited to the interval before the time at which a disturbance emitted from the tip at $t = 0$ has travelled to the boundary and back to the tip (t_d).

FORMAL SOLUTION

Let us first consider the case when the tip stops momentaneously at $t = 0$ (problem B). We then obtain some time-dependent stress-distribution, say $\tau_{yz} = \tau^{st}(x, t)$ in front of the tip in the crack plane (Fig. 1b). Of interest is the stress-intensity factor $K^{st}(t)$. This can be found in [10]. After a simple change of integration variable we obtain

$$K^{st}(t) = \mu\left(1 + \frac{V}{C}\right)^{1/2}\left(\frac{2}{\pi}\right)^{1/2}\int_{\eta=0}^{-Vt}(Vt+\eta)^{-1/2}\frac{\partial w^s}{\partial \eta}\,d\eta. \tag{5}$$

μ is the shear modulus and C the velocity of the equivoluminal waves.

Consider now the following boundary-value problem (C) for a half-space with radiation conditions at infinity (Fig. 1c). The medium is assumed to be at rest for $t \leq 0$.

$$\tau_{yz} = -H(x)\tau^{st}(x, t) \qquad x < a(t) \tag{6}$$

$$w = 0 \qquad x > a(t). \tag{7}$$

$a(t)$ is an arbitrary time-function and H denotes the unit step function.

It is obvious that if we superpose problem (C) onto problem (B), we will obtain the solution to the desired problem. The stresses for the two problems on the segment $x < a(t)$ cancel each other and the displacement condition for $x > a(t)$ is satisfied. The latter problem (C) is however precisely the one considered by Kostrov[1]. In Kostrov's paper an explicit expression for the stress-intensity factor is given. Inserting the particular stress-distribution (6) we obtain.

$$K(t) = \left(1 - \frac{\dot{a}(t)}{C}\right)^{1/2}\left(\frac{2}{\pi}\right)^{1/2}\int_{x=a(t)-Ct}^{a(t)} -\tau^{st}\left(x, t - \frac{a(t)}{C} + \frac{x}{C}\right)(a(t)-x)^{-1/2}\,dx \tag{8}$$

if K is defined as

$$K = \lim_{x\to +a(t)} [2\pi(x - a(t))]^{1/2}\tau_{yz}(x, t). \tag{9}$$

Equation (8) formally solves our problem for $t < t_d$. If $\tau^{s''}(x, t)$ was known, the integral could be solved. We will here take a different route, but a necessary result for the following analysis can be deduced from (8). The integral depends only of the current position $a(t)$ of the tip and on $\tau^{s''}(x, t)$. In other words, $K(t)$ does in no way depend on how the tip has reached its position at a certain time-instant. We will now distinguish between the two cases $a(t) < Vt$ and $a(t) > Vt$ at a particular time instant t. These cases will be treated somewhat differently.

SOLUTION FOR THE CASE $a(t) < Vt$

As discussed above, the integral in (8) will have the same value for any growth history having the property that the tip's position is $x = a(t)$ at the considered time-instant. Let us therefore construct a particular growth history with this property (Fig. 2).

$$\dot{a}(t_1) = V \quad \text{for} \quad t_1 < t_0 = \frac{a(t)}{V} < t \tag{10}$$

$$\dot{a}(t_1) = 0 \quad \text{for} \quad t_1 \geq t_0. \tag{11}$$

This is just the case when the crack continues to move steadily and instead stops at the time t_0. $K(t)$ is then given by (5) with t replaced by the time-difference $t - t_0$. We then immediately obtain K for any other growth history by simply multiplying with the velocity-dependent factor of (8).

$$K(t) = \mu\left(1 + \frac{V}{C}\right)^{1/2}\left(1 - \frac{\dot{a}(t)}{C}\right)^{1/2}\left(\frac{2}{\pi}\right)^{1/2}\int_{\eta=0}^{\Delta(t)} (-\Delta(t) + \eta)^{-1/2}\frac{\partial w^s}{\partial \eta}\,d\eta \tag{12}$$

with

$$\Delta(t) = a(t) - Vt. \tag{13}$$

SOLUTION FOR THE CASE $a(t) > Vt$

Consider another boundary-value problem (D) for the half-space (Fig. 1d).

$$\tau_{yz} = -H(x - Vt)\tau^s(x - Vt) \qquad 0 < x < a(t)$$
$$w = 0 \qquad a(t) < x.$$

Superpose problem (D) onto the steady-state problem (A). In analogy with the earlier discussion, this solves the desired problem. Insertion of (14) into (8) and making a simple transformation of integration variable lead to eqn (16).

$$K(t) = \left(1 - \frac{V}{C}\right)^{-1/2}\left(1 - \frac{\dot{a}(t)}{C}\right)^{1/2}\left(\frac{2}{\pi}\right)^{1/2}\int_{\eta=0}^{\Delta(t)} (\Delta(t) - \eta)^{-1/2}\tau^s(\eta)\,d\eta \tag{16}$$

$\Delta(t)$ is defined by (13).

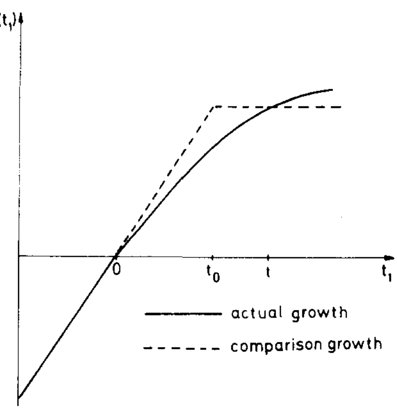

Fig. 2. Actual and comparison crack growth history, $a(t) < V(t)$.

The solution of the title problem is thus given by eqns (12) and (16). The integrals in these equations depend only of how far the crack has extended relative to its position had it continued to move steadily. For any particular problem these integrals can always be evaluated at least by aid of numerical methods. We can thus summarize the results as

$$K(t) = \left(1 - \frac{\dot{a}(t)}{C}\right)^{1/2} f(\Delta(t); V) \tag{17}$$

where the function f can be calculated once and for all for a specific problem.

In the particular case when $\Delta(t)$ tends to zero, only the singular parts of w^s and τ^s give contributions to the integrals. One has for the singular parts of these functions (see [11]).

$$w^s \to \frac{K^D}{\mu}(1 - V^2/C^2)^{-1/2}\left(-\frac{2\eta}{\pi}\right)^{1/2} \quad \text{as} \quad \eta \to -0 \tag{18}$$

$$\tau^s \to K^D(2\pi\eta)^{-1/2} \quad \text{as} \quad \eta \to +0. \tag{19}$$

K^D is here the stress-intensity factor for the steadily moving crack. Insertion of these expressions into (12) and (16) yields as it should the common result

$$K(t) = \left(\frac{C - \dot{a}(t)}{C - V}\right)^{1/2} K^D. \tag{20}$$

APPLICATION TO ARREST PROBLEMS

It has in the treatment been assumed that the body is homogenous as what regards the elastic properties. Velocity changes of the kind assumed here can thus only occur if the fracture properties of the material vary. Suppose e.g. that the body is composed of two regions with different fracture properties, say $K_c^{\mathrm{I}}(\dot{a})$ and $K_c^{\mathrm{II}}(\dot{a})$ in regions I and II respectively.

Let the crack move steadily with velocity V in region I. As the tip hits region II a sudden velocity change will occur if $K_c^{\mathrm{II}}(V)$ is different from $K_c^{\mathrm{I}}(V)$.

During the steady phase of motion K^D obviously must equal $K_c^{\mathrm{I}}(V)$. Applying eqn (19) we thus obtain

$$K_c^{\mathrm{I}}(V)(C - V)^{-1/2} = K_c^{\mathrm{II}}(\dot{a}(+0))(C - \dot{a}(+0))^{-1/2}. \tag{21}$$

From this equation $\dot{a}(+0)$ can be determined if the remaining quantities are known. The equation for the subsequent motion can with aid of (17) be written

$$K_c^{\mathrm{II}}(\dot{a}(t))\left(1 - \frac{\dot{a}(t)}{C}\right)^{-1/2} = f(a(t) - Vt; V). \tag{22}$$

This non-linear, first order equation has in general to be solved numerically with the initial condition given by (21). This can be accomplished by standard numerical methods.

A practical way of crack arrest design is to insert bands of tougher material. The question is in that case if the crack stops before reaching the parent material again. A solution of (22) which gives zero velocity at some time-instant before t_d, does not necessarily mean that the crack ceases to move completely. As was observed in [10], K may oscillate after a stopping and in some cases reinitiation of growth may occur.

AN EXAMPLE

Consider the same example as in [10], i.e. a clamped infinite strip bisected by a running semi-infinite crack (Fig. 3). The boundaries $y = \pm h$ are displaced an amount $\pm w_0$ respectively.

The steady-state problem was solved in [12] and [13]. The expressions for the necessary quantities are given by

$$\frac{\partial w^s}{\partial \eta} = \frac{w_0}{\beta h}\left(e^{(-\pi\eta/\beta h)} - 1\right)^{-1/2}, \quad \eta < 0 \tag{23}$$

$$\tau^s = \mu\frac{w_0}{h}(1 - e^{(-\pi h/\beta h)})^{-1/2}, \quad \eta > 0 \tag{24}$$

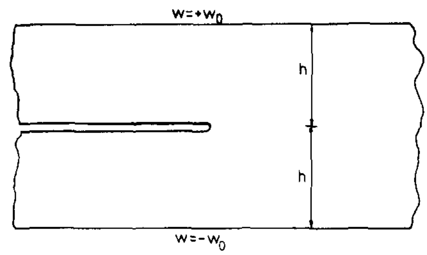

Fig. 3. The strip problem.

where

$$\beta = (1 - V^2/C^2)^{1/2}.$$

Furthermore we have from [13]

$$K^D = \mu w_0 \left(\frac{2\beta}{h}\right)^{1/2} = K_{st}(\beta)^{1/2} \tag{26}$$

K_{st} is the static stress-intensity factor.

Insertion of these expressions into eqns (12) and (16) gives the stress-intensity factor $K(t)$.

$$K(t) = \left(\frac{C - \dot{a}(t)}{C - V}\right)^{1/2} K^D \cdot g\left(\frac{\Delta(t)}{\beta h}\right) \tag{27}$$

where

$$g\left(\frac{\Delta(t)}{\beta h}\right) = \frac{1}{\sqrt{\pi}} \operatorname{sgn}(\Delta(t)) \int_0^{\Delta(t)/\beta h} \left(\left(\frac{\Delta(t)}{\beta h} - \zeta\right)(1 - e^{-\pi\zeta})\right)^{-1/2} d\zeta. \tag{28}$$

This integral was calculated numerically and the resulting values are shown in Fig. 4. Using this graph and eqn (27), $K(t)$ can be found for any motion. In Fig. 5 some examples of growth

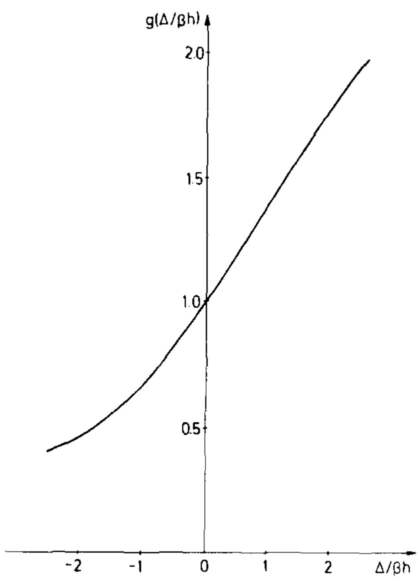

Fig. 4. The function $g(\Delta/\beta h)$.

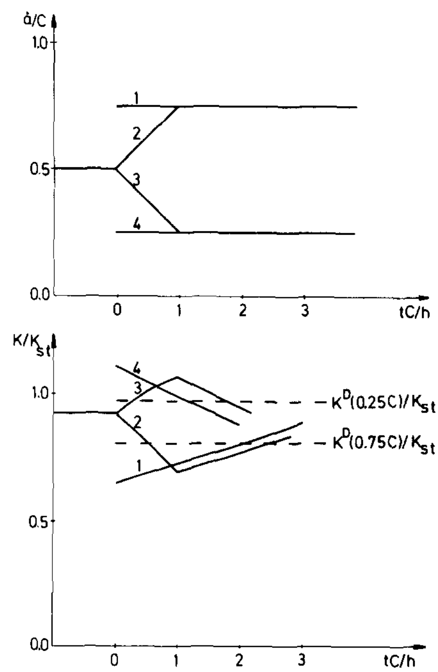

Fig. 5. Examples of velocity variations and resulting stress-intensity factors.

histories are shown together with the resulting values of $K(t)$. The steady initial velocity is in all cases $0.5\,C$. The curves have been drawn up to the time t_d, which is given by the following relation.

$$(a(t_d))^2+4h^2=(Ct_d)^2. \tag{29}$$

For case 1, $K(t)$ first decreases discontinuously and then increases almost linearly to approximately the steady-state value of the new velocity. In case 2 the same pattern is observed, but the initial decrease is in this case continuous. The other two cases have a similar structure but with the sign of the changes in K reversed.

DISCUSSION

The strict applicability of the given results is limited. We have firstly that the analysis is valid only for mode III problems. It is in the authors opinion more than likely that a similar analysis can be carried out for mode I problems, judging from the in other cases found analogues between elastic mode I and III problems. Presumably the analysis will be considerably more complicated (compare e.g. [1–4] to [5–8]).

The second limitation is that a steady state growth must precede the velocity changes. The third is the mentioned restriction on the time range. It does not seem likely that general analytical methods can be developed so as to remove these limitations. For particular problems progress can possibly be made.

One relation that may have a more general applicability is eqn (20). Since only the singular field contributes and this is same whether the crack moves uniformly or not[14], it is possible that this relation holds for a discontinuous velocity change under any conditions. K^D should then be identified with the momentaneous value of K before the velocity jump. This simple reasoning is however not a proof, but the matter seems to merit a further investigation.

In view of the discussed limitations it is unlikely that the derived results are of much practical value for crack arrest design. On the contrary, they can be extremely valuable in experimental work where the experiment can be constructed so as to at least approximately fulfill the assumptions. That is, it is possible to evaluate a K_c vs a relation by using the strip specimen.

Solutions of the present type can also be used as test examples for numerical analysis. It is to be expected that the application of FEM to dynamic crack problems will be increasingly popular. It is then necessary to have reference solutions in order to check the numerical methods.

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